

On volumes determined by subsets of Euclidean space

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ABSTRACT. Given $E \subset \mathbb{R}^d$, define the *volume set* of E , $\mathcal{V}(E) = \{\det(x^1, x^2, \dots, x^d) : x^j \in E\}$. In \mathbb{R}^3 , we prove that $\mathcal{V}(E)$ has positive Lebesgue measure if either the Hausdorff dimension of $E \subset \mathbb{R}^3$ is greater than $\frac{13}{5}$, or E is a product set of the form $E = B_1 \times B_2 \times B_3$ with $B_j \subset \mathbb{R}$, $\dim_{\mathcal{H}}(B_j) > \frac{2}{3}$, $j = 1, 2, 3$. We show that the same conclusion holds for $\mathcal{V}(E)$ of Salem subsets $E \subset \mathbb{R}^d$ with $\dim_{\mathcal{H}}(E) > d - 1$, and give applications to discrete combinatorial geometry.

1. Introduction

A large class of Erdős type problems in geometric combinatorics ask whether a large set of points in Euclidean space determines a suitable large sets of geometric relations, configurations or objects. For example, the classical Erdős distance problem asks whether N points in \mathbb{R}^d , $d \geq 2$, determines $\gtrsim N^{\frac{2}{d}}$ distinct distances. See, for example [1, 13, 15, 16, 17] and the references there for thorough descriptions of these types of problems and recent results. (Here, and throughout, $X \lesssim Y$, if the controlling parameter is N , means that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $X \leq C_\epsilon N^\epsilon Y$, while $X \lesssim Y$ means $X \leq CY$ with C independent of N .)

Continuous variants of Erdős type geometric problems have also received much attention in recent decades. Perhaps the best known of these is the Falconer distance problem [7], which asks whether the one-dimensional Lebesgue measure of the distance set $\{|x - y| : x, y \in E\}$ is positive, provided that the Hausdorff dimension, $\dim_{\mathcal{H}}(E)$, of $E \subset \mathbb{R}^d$, $d \geq 2$, is greater than $\frac{d}{2}$. See [19, 4] for the best currently known results on this problem. Also see [5] for the closely related problem of finite point configurations.

In this paper we study the sets of *volumes* determined by sets $E \subset \mathbb{R}^d$. Given d vectors x^1, x^2, \dots, x^d in \mathbb{R}^d , $d \geq 2$, let $\det(x^1, \dots, x^d)$ denote the determinant of the matrix whose j th column is x^j . For $E \subset [0, 1]^d$, define the *volume set* of E ,

$$\mathcal{V}(E) := \{\det(x^1, x^2, \dots, x^d) \in \mathbb{R} : x^j \in E\}.$$

A problem is to find the optimal threshold $s_{\mathcal{V}}(d)$ such that if the Hausdorff dimension of $E \subset [0, 1]^d$ is greater than $s_{\mathcal{V}}(d)$, then the Lebesgue measure of $\mathcal{V}(E)$, denoted by $\mathcal{L}^1(\mathcal{V}(E))$, is positive. Letting E be a $(d - 1)$ -dimensional hyperplane shows that one must take $s_{\mathcal{V}}(d) \geq d - 1$.

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A result due to Erdoğan, Hart and the second author [6] shows that $s_{\mathcal{V}}(2) \leq \frac{3}{2}$. More generally, they prove the following result.

Theorem 1.1. [6] *For $d \geq 2$, let $E, F \subset [0, 1]^d$ and Q be a non-degenerate bilinear form on \mathbb{R}^d . Suppose that $\dim_{\mathcal{H}}(E) + \dim_{\mathcal{H}}(F) > d + 1$. Then $\mathcal{L}^1(\{Q(x, y) : x \in E, y \in F\}) > 0$.*

Thus, if $E \subset \mathbb{R}^2$ with $\dim_{\mathcal{H}}(E) > \frac{3}{2}$, then $\mathcal{L}^1(\mathcal{V}(E)) > 0$.

Our four main results are the following.

Theorem 1.2. *Let $E \subset [0, 1]^3$, with $\dim_{\mathcal{H}}(E) > \frac{13}{5}$. Then $\mathcal{L}^1(\mathcal{V}(E)) > 0$.*

In view of Thm. 1.1 above, Thm. 1.2 may be reduced to studying the Hausdorff dimension of the set of $(d - 1)$ -vectors of E ,

$$(1.1) \quad \Lambda(E) := \{*(x^1 \wedge x^2 \wedge \cdots \wedge x^{d-1}) : x^j \in E, 1 \leq j \leq d - 1\},$$

where $*$ is the Hodge star operator, $*$: $\Lambda^{d-1}\mathbb{R}^d \rightarrow \mathbb{R}^d$. Since

$$(1.2) \quad \det(x^1, \dots, x^d) = \pm x^d \cdot *(x^1 \wedge x^2 \wedge \cdots \wedge x^{d-1}),$$

establishing Thm. 1.2 reduces to proving the following result.

Theorem 1.3. *Let $E \subset [0, 1]^3$ and suppose that $\dim_{\mathcal{H}}(E) > \frac{13}{5}$. Then*

$$(1.3) \quad \dim_{\mathcal{H}}(E) + \dim_{\mathcal{H}}(\Lambda(E)) > 4.$$

If one could strengthen (1.3) to say that the left hand side is greater than 3, then Thm. 1.2 would be improved to the $\dim_{\mathcal{H}}(E) > 2$. This would be optimal because, as noted above, the conclusion of Thm. 1.2 does not in general hold if $\dim_{\mathcal{H}}(E) \leq d - 1$.

Remark 1.4. In the context of three dimensional vector spaces over finite fields, the analogue of Theorem 1.2 is completely resolved, with a sharp exponent using sum-product technology; see [2]. However, sum-product issues are generally believed to be more difficult in the continuous setting.

It is possible to obtain a better exponent than in Thm. 1.3, and thus Thm. 1.2, if the set E has a special form or satisfies Fourier decay conditions. We first consider the situation where the set under consideration is a Cartesian product of subsets of the real line.

Theorem 1.5. *Suppose that $E = B_1 \times B_2 \times B_3$, where $B_j \subset [0, 1]$ satisfy $\dim_{\mathcal{H}}(B_j) > \frac{2}{3}$, $j = 1, 2, 3$. Then $\mathcal{L}^1(\mathcal{V}(E)) > 0$.*

Note that if each B_j is Ahlfors-David regular (see e.g. [14]), the assumption that the Hausdorff dimension of each B_j is greater than $\frac{2}{3}$ is equivalent to the assumption that the Hausdorff dimension of $E = B_1 \times B_2 \times B_3$ is greater than 2. See, e.g., [8, 14].

Alternatively, one can also consider the situation in higher dimensions, when set $E \subset \mathbb{R}^d$, $d \geq 2$, supports a measure that has good Fourier decay properties. Recall that $E \subset [0, 1]^d$, $d \geq 2$, is *Salem* if there exists a probability measure μ supported on E such that

$$(1.4) \quad |\widehat{\mu}(\xi)| \lesssim |\xi|^{-\frac{s}{2}}, \quad |\xi| \rightarrow \infty,$$

where s denotes the Hausdorff dimension of E and

$$\widehat{\mu}(\xi) = \int e^{-2\pi i x \cdot \xi} d\mu(x)$$

is the Fourier transform of μ . We prove

Theorem 1.6. *Suppose $E \subset \mathbb{R}^d$ is Salem and $s := \dim_{\mathcal{H}}(E) > d - 1$. Then $\mathcal{L}^1(\mathcal{V}(E)) > 0$.*

We note that while one cannot do better than the exponent $d-1$, as the example of a hyperplane shows, it is possible that this might not be the case for Salem sets. It is conceivable that the conclusion of Theorem 1.6 holds if the Hausdorff dimension of E is merely greater than one.

2. Proof of Theorem 1.6

Let ψ be a smooth cut-off function on \mathbb{R}^d , $\equiv 1$ in the unit ball and supported in the ball of radius \sqrt{d} . Then

$$\begin{aligned}
 & (\mu \times \mu \times \cdots \times \mu)\{(x^1, x^2, \dots, x^d) : t \leq \det(x^1, x^2, \dots, x^d) \leq t + \epsilon\} \\
 & \approx \int \int \cdots \int \psi\left(\frac{\det(x^1, \dots, x^d) - t}{\epsilon}\right) d\mu(x^1) \dots d\mu(x^d) \\
 (2.1) \quad & = \int \int \cdots \int \int e^{2\pi i \frac{\lambda}{\epsilon} \det(x^1, \dots, x^d)} e^{-2\pi i \frac{t\lambda}{\epsilon}} d\mu(x^1) \dots d\mu(x^d) \widehat{\psi}(\lambda) d\lambda.
 \end{aligned}$$

Using (1.2), it follows that (2.1) equals

$$\begin{aligned}
 & \epsilon \int \cdots \int \widehat{\mu}(\lambda(x^1 \wedge \cdots \wedge x^{d-1})) e^{-2\pi i t \lambda} d\mu(x^1) \dots d\mu(x^{d-1}) \widehat{\psi}(\epsilon \lambda) d\lambda \\
 & = \epsilon \int \cdots \int_{|x^1 \wedge \cdots \wedge x^{d-1}| \leq \lambda^{-1}} \widehat{\mu}(\lambda(x^1 \wedge \cdots \wedge x^{d-1})) e^{-2\pi i t \lambda} d\mu(x^1) \dots d\mu(x^{d-1}) \widehat{\psi}(\epsilon \lambda) d\lambda \\
 & + \epsilon \int \cdots \int_{|x^1 \wedge \cdots \wedge x^{d-1}| > \lambda^{-1}} \widehat{\mu}(\lambda(x^1 \wedge \cdots \wedge x^{d-1})) e^{-2\pi i t \lambda} d\mu(x^1) \dots d\mu(x^{d-1}) \widehat{\psi}(\epsilon \lambda) d\lambda := I + II.
 \end{aligned}$$

Since $|\widehat{\mu}(\xi)| \leq 1$, we have

$$I \lesssim \epsilon \int (\mu \times \cdots \times \mu)\{(x^1, \dots, x^{d-1}) : |x^1 \wedge \cdots \wedge x^{d-1}| \leq \lambda^{-1}\} |\widehat{\psi}(\epsilon \lambda)| d\lambda.$$

Lemma 2.1. *With the notation above,*

$$(2.2) \quad (\mu \times \cdots \times \mu)\{(x^1, \dots, x^{d-1}) : |x^1 \wedge \cdots \wedge x^{d-1}| \leq \lambda^{-1}\} \lesssim \min\{1, \lambda^{d-2-s}\}.$$

To prove the lemma, start by noting that one has

$$|x^1 \wedge x^2 \wedge \cdots \wedge x^{d-1}| = |x^1| \cdot |x^2 \wedge \cdots \wedge x^{d-1}| \cdot \sin(\theta),$$

where θ is the angle between x^1 and the $(d-2)$ -plane spanned Π by x^2, \dots, x^{d-1} . Localize to where $|x^2 \wedge \cdots \wedge x^{d-1}| \approx 2^{-j}$. By induction, the measure of this set is $\lesssim 2^{j(d-3-s)}$. Since x^1 is contained in a $2^j \lambda^{-1}$ -tubular nhoo of Π , it follows that

$$\begin{aligned}
 (\mu \times \cdots \times \mu)\{(x^1, \dots, x^{d-1}) : |x^1 \wedge \cdots \wedge x^{d-1}| \leq \lambda^{-1}\} & \lesssim (2^j \lambda^{-1})^{-(d-2)} (2^j \lambda^{-1})^s (2^{-j})^{s-(d-3)} \\
 & = \lambda^{d-2-s} 2^{-j}.
 \end{aligned}$$

It follows that the left hand side of (2.2) is bounded by a constant multiple of

$$\sum_j \lambda^{d-2-s} 2^{-j} \lesssim \lambda^{d-2-s},$$

completing the proof of Lemma 2.1.

It follows from Lemma 2.1 that

$$I \lesssim \int \min\{1, \lambda^{d-2-s}\} |\widehat{\psi}(\epsilon\lambda)| d\lambda,$$

which is $\lesssim 1$ if $s > d - 1$. By the Salem property (1.4),

$$\begin{aligned} II &\lesssim \epsilon \int \dots \int \lambda^{-\frac{s}{2}} |x^1 \wedge \dots \wedge x^{d-1}|^{-\frac{s}{2}} d\mu(x^1) d\mu(x^2) \dots d\mu(x^{d-1}) |\widehat{\psi}(\epsilon\lambda)| d\lambda \\ &\approx \sum_{j \leq \log_2(\lambda)} 2^{\frac{js}{2}} \epsilon \int \lambda^{-\frac{s}{2}} \mu \times \dots \times \mu\{(x^1, \dots, x^{d-1}) : |x^1 \wedge \dots \wedge x^{d-1}| \approx 2^{-j}\} |\widehat{\psi}(\epsilon\lambda)| d\lambda. \end{aligned}$$

Using Lemma 2.1, one sees that this is this is

$$\approx \int \sum_{j \leq \log_2(\lambda)} 2^{j(d-2-s)} 2^{\frac{js}{2}} \lambda^{-\frac{s}{2}} |\widehat{\psi}(\epsilon\lambda)| d\lambda \lesssim \int_1^\infty \lambda^{d-2-s} |\widehat{\psi}(\epsilon\lambda)| d\lambda,$$

which is bounded independently of ϵ if $s > d - 1$, as desired. This finishes the proof of Thm. 1.6.

3. Proof of Theorem 1.5

We shall need the following consequence of Theorem 1.0.3 in [6].

Theorem 3.1. [6] *Let $A_1, A_2, A_3, A_4 \subset [0, 1]$, each with $\dim_{\mathcal{H}}(A_j) > \frac{2}{3}$. Then*

$$(3.1) \quad \mathcal{L}^1(\{a_1 a_2 + a_3 a_4 : a_j \in A_j\}) > 0.$$

To apply Thm. 3.1, consider

$$(3.2) \quad \det(x, y, z) = y_1(x_3 z_2 - x_2 z_3) + y_2(x_1 z_3 - x_3 z_1) + y_3(-x_1 z_2 - x_2 z_1).$$

Fix $x_1, x_2, x_3, y_3, z_3 \in A$, all $\neq 0$, and let

$$w_1 = x_3 z_2 - x_2 z_3, \quad w_2 = x_1 z_3 - x_3 z_1.$$

Observe that

$$-x_1 z_2 - x_2 z_1 = -\frac{x_1}{x_3} w_1 - \frac{x_2}{x_3} w_2.$$

It follows that the expression in (3.2) equals

$$y_1 w_1 + y_2 w_2 - \frac{x_1}{x_3} w_1 - \frac{x_2}{x_3} w_2 = w_1 \left(y_1 - \frac{x_1}{x_3} \right) + w_2 \left(y_2 - \frac{x_2}{x_3} \right).$$

Let

$$A_1 = x_3 B_2 - x_2 B_3, \quad A_2 = x_1 B_3 - x_3 B_1, \quad A_3 = B_1 - \frac{x_1}{x_3}, \quad A_4 = B_2 - \frac{x_2}{x_3}.$$

From the assumption that $\dim_{\mathcal{H}}(A) > \frac{2}{3}$, it follows that each $\dim_{\mathcal{H}}(A_j)$ is also $> \frac{2}{3}$. By the above calculation, one has

$$\{a_1a_2 + a_3a_4 : a_j \in A_j\} \subset \mathcal{V}(B_1 \times B_2 \times B_3),$$

and so the conclusion of Theorem 1.5 follows by Theorem 3.1.

4. Proof of Theorem 1.3

As noted in the introduction, in view of Thm. 1.1, it suffices to prove Thm. 1.3. For clarity and possible future use, we begin the analysis in \mathbb{R}^d , specializing to $d = 3$ later on. To this end, define a natural measure on the set of wedge products by the relation

$$\int f(z) d\Lambda(z) = \int \dots \int f(x^1 \wedge \dots \wedge x^{d-1}) d\mu(x^1) \dots d\mu(x^{d-1}).$$

It follows that

$$\widehat{\Lambda}(\xi) = \int \dots \int e^{-2\pi i \xi \cdot (x^1 \wedge \dots \wedge x^{d-1})} d\mu(x^1) \dots d\mu(x^{d-1})$$

and thus

$$(4.1) \quad \int \left| \widehat{\Lambda}(\xi) \right|^2 \psi\left(\frac{\xi}{R}\right) d\xi = R^d \int \int \widehat{\psi}(R(x^1 \wedge \dots \wedge x^{d-1} - y^1 \wedge \dots \wedge y^{d-1})) d\mu(x^1) \dots d\mu(x^{d-1}) d\mu(y^1) \dots d\mu(y^{d-1}).$$

Since $\widehat{\psi}$ is rapidly decaying, it suffices to estimate

$$(\mu \times \dots \times \mu) \{ (x^1, \dots, x^{d-1}, y^1, \dots, y^{d-1}) : |x^1 \wedge \dots \wedge x^{d-1} - y^1 \wedge \dots \wedge y^{d-1}| \leq R^{-1} \}.$$

Let $B_r^d(x)$ be the d -dimensional ball of radius r centered at x ; if $x = 0$, the center is suppressed. A_r^d will denote a d -dimensional annulus of inner/outer radii $\frac{r}{2}$ and r , not necessarily centered at the origin. For both, d is the ambient dimension if not included in the notation; lower dimensional balls and annuli in \mathbb{R}^d are denoted using the superscript. Denote $(d-1)$ -tuples of vectors in \mathbb{R}^d by $\vec{x} = (x^1, \dots, x^{d-1})$, and then, as above, $\hat{x} = *(x^1 \wedge \dots \wedge x^{d-1})$, viewed as a vector.

Let μ be a Frostman measure on the s -dimensional $E \subset A_1^d$ [14]. To control (4.1), we want to estimate $R^d \cdot \mu^{2d-2}(F^R)$, where

$$F^R := \{(\vec{x}, \vec{y}) : |\hat{x} - \hat{y}| < 1/R\}.$$

Start by decomposing

$$B_1^d = B_{1/R}(0) \cup \bigcup_{i=0}^{\log_2 R} \bigcup_{j=0}^{(R/2^i)^d} B_{1/R}(z_j^{(i)}),$$

where $\{z_j^{(i)}\}_j$ is a $(1/R)$ -net of points in the dyadic shell $A_{2^{-i-1}, 2^{-i}}$. Then we can write

$$\begin{aligned} F^R &= \{(\vec{x}, \vec{y}) : \hat{x}, \hat{y} \in B_{1/R}\} \cup \bigcup_i \bigcup_j \{(\vec{x}, \vec{y}) : \hat{x}, \hat{y} \in B_{1/R}(z_j^{(i)})\} \\ &\subset (G_0 \times G_0) \cup \bigcup_i \bigcup_j G_j^{(i)} \times G_j^{(i)}, \end{aligned}$$

where

$$G_0 = \{\vec{x} : \hat{x} \in B_{1/R}\}, \quad G_j^{(i)} = \{\vec{x} : \hat{x} \in B_{1/R}(z_j^{(i)})\};$$

in terms of measure, the $G_j^{(i)}$ essentially only depend on i and we refer to them generically as $G^{(i)}$. With all this, one has

$$(4.2) \quad R^d \mu^{2d-2}(F^R) \lesssim R^d \left(\mu^{d-1}(G_0)^2 + \sum_{i=0}^{\log_2 R} \left(\frac{R}{2^i}\right)^d \cdot \left(\mu^{d-1}(G^{(i)})\right)^2 \right),$$

where $\mu^k := \mu \times \cdots \times \mu$, and we want to estimate the terms on the right hand side.

We now restrict ourselves to three dimensions ($d = 3$).

To estimate the G_0 term, note that $x^1 \in E$ is arbitrary, contributing $\mu \lesssim C$. For x^1 fixed, $x^2 \in (1/R) \times (1/R) \times 1$ tube, which we cover with R $(1/R)$ -balls, giving a μ contribution $\lesssim R \cdot R^{-s} = R^{1-s}$. Thus, $\mu^2(G_0) \lesssim R^{1-s}$ and hence $\mu^4(G_0 \times G_0) \lesssim R^{2-2s}$.

To estimate the $G^{(i)}$ term, start by noting that, with $z_0 := z_j^{(i)}$ fixed, x^1 must be in a thin nhood of thickness $2^i/R$ in \mathbb{R}^3 of an annulus $A_1^2 \subset z_0^\perp$; call such a set a washer. Covering this with $(R/2^i)^2$ balls of radius $2^i/R$ gives a $\mu \lesssim (R/2^i)^{2-s}$. For each such x^1 fixed, x^2 must satisfy two constraints: (i) It has to be in the same washer as x^1 ; and (2) It has to make an angle of $\sim 2^{-i}$ with x^1 and vary in the radial direction by $\leq 2^i/R$. There are thus two cases,

$$(a) \ 1 \leq 2^i \leq R^{\frac{1}{2}}; \text{ and } (b) \ R^{\frac{1}{2}} \leq 2^i \leq R.$$

In case (a), $2^i/R \leq 2^{-i}$, so x^2 is confined to a $2^{-i} \times (2^i/R) \times (2^i/R)$ -tube, which is covered by $R/2^{2i}$ $(2^i/R)$ -balls, giving

$$\mu \lesssim (R/2^{2i})(2^i/R)^s = 2^{(s-2)i} R^{1-s}.$$

Multiplying by the μ measure in x^1 gives

$$\mu^2(G^{(i)}) \lesssim 2^{(2s-4)i} R^{3-2s};$$

squaring this and multiplying by the prefactor $(R/2^i)^d = R^3 2^{-3i}$ gives

$$2^{(4s-11)i} R^{9-4s}, \quad 1 \leq 2^i \leq R^{\frac{1}{2}},$$

which, since $4s - 11 < 0$ for the s of interest, takes its largest value for $2^i = 1$, corresponding to generic configurations and yielding the estimate R^{9-4s} (before inclusion of the pre-prefactor of $R^d = R^3$ in (4.2)).

In case (b), $2^i/R \geq 2^{-i}$ and so x^2 is confined to a $2^{-i} \times 2^{-i} \times (2^i/R)$ -tube, which is covered by $2^{2i}/R$ (2^{-i}) -balls, giving

$$\mu \leq (2^{2i}/R) \cdot 2^{-si} = 2^{(2-s)i} R^{-1}.$$

Multiplying by the μ measure in x^1 , squaring and multiplying by the $R^3 2^{-3i}$ gives $2^{-3i} R^{5-2s}$ in the range $R^{\frac{1}{2}} \leq 2^i \leq R$. The largest value is for $2^i = R^{\frac{1}{2}}$, namely $R^{\frac{7}{2}-2s}$, but this is smaller than the R^{9-4s} , as is the G_0 term.

Thus, we estimate (4.2) by $R^3 \cdot R^{9-4s} = R^{12-4s}$. It follows that the expression in (4.1) is $\lesssim R^{3-(4s-9)}$, which implies that the Hausdorff dimension of the set $\Lambda(E)$ is greater than or equal to $4s-9$. To make use of Thm. 1.1, we need $s + (4s-9) > 4$, which holds if $s > \frac{13}{5}$. This completes the proof of Thm. 1.3 and thus of Thm. 1.2.

5. Applications to discrete geometry

Definition 5.1. [12] Let P be a set of n points contained in $[0, 1]^2$. Define the measure

$$(5.1) \quad d\mu_P^s(x) = n^{-1} \cdot n^{\frac{d}{s}} \cdot \sum_{p \in P} \chi_{B_{n^{-\frac{1}{s}}}}(x) dx,$$

where $\chi_{B_{n^{-\frac{1}{s}}}}(x)$ is the characteristic function of the ball of radius $n^{-\frac{1}{s}}$ centered at p . One says that P is *s-adaptable* if

$$I_s(\mu_P) = \int \int |x - y|^{-s} d\mu_P^s(x) d\mu_P^s(y) < \infty.$$

This is equivalent to the statement

$$(5.2) \quad n^{-2} \sum_{p \neq p' \in P} |p - p'|^{-s} \lesssim 1.$$

To understand this condition in clearer geometric terms, suppose that P comes from a 1-separated set A , scaled down by its diameter. Then the condition (5.2) takes the form

$$(5.3) \quad n^{-2} \sum_{a \neq a' \in A} |a - a'|^{-s} \lesssim (\text{diameter}(A))^{-s}.$$

Thus, P is *s-adaptable* if it is a scaled 1-separated set where the expected value of the distance between two points raised to the power $-s$ is comparable to the value of the diameter raised to the power of $-s$. This basically means that clustering within P is not allowed to be too severe.

In more technical terms, *s-adaptability* means that a discrete point set P can be thickened into a set which is uniformly *s-dimensional* in the sense that its energy integral of order s is finite. Unfortunately, it is shown in [12] that there exist finite point sets which are not *s-adaptable* for certain ranges of the parameter s . The point is that the notion of Hausdorff dimension is much more subtle than the simple “size” estimate. However, many natural classes of sets are *s-adaptable*. For example, homogeneous sets studied by Solymosi and Vu [18] and others are *s-adaptable* for all $0 < s < d$. See also [10] where *s-adaptability* of homogeneous sets is used to extract discrete incidence theorems from Fourier-type bounds.

The following argument is a variant of the conversion mechanism developed in [11, 9]. Suppose that one knew that $\mathcal{L}^1(\mathcal{V}(E)) > 0$ whenever $\dim_{\mathcal{H}}(E) > s_{\mathcal{V}}$. Let P be an *s-adaptable* set with $s \in (s_0, d]$, and E denote the support of $d\mu_P^s$ above. It follows that

$$1 \lesssim \mathcal{L}^1(\mathcal{V}(E)) \lesssim n^{-\frac{1}{s}} \cdot \#P,$$

and one can conclude that

$$\#\mathcal{V}(E) \gtrsim n^{\frac{1}{s}} \gtrsim n^{\frac{1}{s_0}},$$

which yields the following theorem.

Theorem 5.2. *Let P be a $\frac{13}{5}$ -adaptable subset of \mathbb{R}^d of size n . Let $\mathcal{V}_\delta(P)$ denote the number of distinct δ -separated volumes determined by P . Then*

$$\#\mathcal{V}_{n^{-\frac{5}{13}}}(P) \gtrsim n^{\frac{5}{13}}.$$

It is important to note that the significance of this result lies in the requirement that the volumes are $n^{-\frac{5}{13}}$ -separated. In the absence of this feature, the exponent is inferior to the sharp result obtained by Dumitrescu and Toth [3], who proved that for any points set $P \subset \mathbb{R}^3$ of size n , $\#\mathcal{V}(P) \gtrsim n$.

Even if one were to improve Thm. 1.2 to the conjecturally sharp $\dim_{\mathcal{H}}(E) > 2$, the resulting modification to Thm. 5.2 would be that $\#\mathcal{V}_{n^{-\frac{1}{2}}}(P) \gtrsim n^{\frac{1}{2}}$, still weaker in terms of the exponent than [3]. The analytic method used here yields a conclusion about $n^{-\frac{1}{s}}$ -separated volumes; we do not know whether it is possible to obtain the sharp exponent using our methods.

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